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On triconnected and cubic plane graphs on given point sets ☆

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ABSTRACT

Let S be a set of $n \geqslant 4$ points in general position in the plane, and let $h < n$ be the number of extreme points of S. We show how to construct a 3-connected plane graph with vertex set S, having $\max\{\lceil 3n/2 \rceil, n+h-1\}$ edges, and we prove that there is no 3-connected plane graph on top of S with a smaller number of edges. In particular, this implies that S admits a 3-connected cubic plane graph if and only if $n \geqslant 4$ is even and $h \leqslant n/2+1$. The same bounds also hold when 3-edge-connectivity is considered. We also give a partial characterization of the point sets in the plane that can be the vertex set of a cubic plane graph.

1. Introduction

1.1. Preliminaries and previous work

A *geometric graph* G is a simple finite graph whose vertex set $V(G)$ is a finite set of points in general position in the plane (i.e., no three of them are collinear), and each edge in $E(G)$ is a closed segment whose endpoints belong to $V(G)$. If $V(G) = S$ we also say that the geometric graph G *is on top of* S, or simply that G *is on* S. A geometric graph is a *plane graph* if no two edges cross. That is, two edges in a plane graph may intersect only at a common endpoint. It is also usual to use the expressions *non-crossing geometric graph* or *crossing-free geometric graph* as synonymous for *plane graph*. A (geometric) graph is *cubic*, if the degree of every vertex is three. A (geometric) graph on at least $k+1$ vertices is *k-connected* if it is connected and it remains connected whenever $k-1$ vertices are removed.

Problems on geometrically embedding planar graphs on given point sets have been attracting attention for almost two decades. Ikebe et al. [11] proved that a tree can always be drawn with a prescribed root, culminating previous results by Perles and by Pach and Törőcsik [17], and a similar result with prescribed degrees is given in [20]. Kaneko and Kano [14] obtained an extension to two trees. The fact that outerplanar graphs are the largest graph class always admitting embeddings was proven by Gritzmann et al. [10]. Several papers have been devoted to the algorithmic counterpart of these results [2,3,13,18]. We refer the reader to the books [4,5] for more details on geometric graphs and on graph drawing algorithms.

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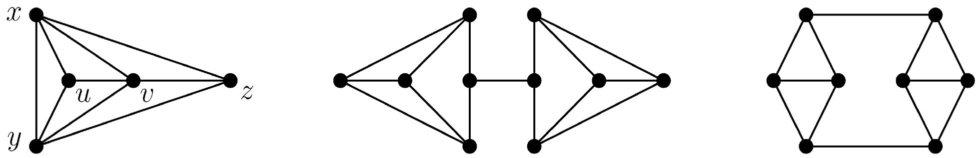

Fig. 1. Left: Starting with xy, yz, zx, vx, vy, vz, vertex u is inserted and joined to x, y and v. The resulting graph remains 3-connected if the edge xy is suppressed. Middle and right: Connected cubic plane graphs which are not 3-connected.

For any set S of n points in general position in the plane it is easy to construct a connected plane graph on top of S, even with the additional requirement that it has minimum possible number of edges, $n-1$. For example, we may take the minimum spanning tree of S or we may connect the points by a path visiting the points of S in lexicographically increasing order, say, of their coordinates. Similarly, it is also not difficult to construct a 2-connected plane graph on top of S with the minimum number, n, of edges: We can construct a *polygonization of S*, i.e., a simple polygon whose vertex set is S. Methods yielding polygonizations were described by Steinhaus and by Gemignani [9,19] and later systematically studied in the field of computational geometry.

On the opposite direction, there are point sets that do not admit any 4-connected plane graph on top of them. Some examples are given by Dey et al. in the paper [7], where they also provide a necessary and sufficient condition for point sets whose convex hull consists of exactly three vertices. However, a general characterization of the sets of points admitting a 4- or 5-connected plane graph is not known [7].

For the case of 3-connectivity this characterization is quite obvious and was described in [7] as well. Let us recall that we say that a point set S is *in convex position* if each point of S is *extreme* (a vertex of the convex hull of S). If S is in convex position, then every plane triangulation of S contains vertices of degree two, therefore it is impossible to get any 3-connected plane graph on top of S. On the contrary, when S is not in convex position, it is easy to check that the following method produces a 3-connected plane graph on S: Let C be the cycle formed by the edges connecting consecutive vertices of the convex hull of S and let $v \in S$ be any point interior to the convex hull; join v to all the vertices in C and then insert iteratively the remaining points. At each step the point being inserted is connected to the three vertices of the triangular face it falls into.

Notice that in general this algorithm does not produce a 3-connected plane graph using as few edges as possible, see Fig. 1 (left). In fact, it always produces a *triangulation of S*, i.e., a plane graph on S with the maximum number of edges, in which all faces are triangles with the only possible exception of the outer face.

In this paper we aim to the minimality of the construction, as was already known for 1- and 2-connectivity, and we describe a polynomial algorithm which, given a point set S not in convex position, finds a 3-connected plane graph on S with the minimum number of edges. Achieving good connectivity by adding as few edges as possible is a classic family of problems in graph theory; we refer the interested reader to the large literature on augmentation problems [1,6,8,12,15,16, 22,23].

Another natural and related problem that we consider here is that of characterizing the point sets that admit a cubic plane graph. Observe that a connected cubic graph on top of S is not necessarily 3-connected, see Fig. 1 (middle and right); therefore, a specific approach is required. The analogous problem of constructing 1- or 2-regular plane graphs is easily solved using a polygonization on S mentioned above — the edges of a simple polygon P on S form a 2-regular plane graph and, if n is even, taking every second segment in P (or in any plane Hamiltonian path on S) gives a 1-regular plane graph on S.

1.2. Results

Throughout the paper, S denotes a set of $n \geqslant 4$ points in general position in the plane, $H = H(S)$ denotes the set of vertices of the convex hull of S, $h = h(S)$ denotes the size of H, and $I = I(S) = S \setminus H$ denotes the set of interior points of S.

Here is our main result:

Theorem 1. *Let S be a set of n points in general position in the plane. Suppose that S is not in convex position. Then there is a 3-connected plane graph on S with $\max\{\lceil 3n/2\rceil, n + h(S) - 1\}$ edges, and it can be found in polynomial time. Moreover, there is no 3-connected plane graph on S with a smaller number of edges.*

Theorem 1 immediately gives the following characterization of sets admitting 3-connected cubic plane graphs:

Corollary 2. *Let S be a set of $n \geqslant 4$ points in general position in the plane. Then there is a 3-connected cubic plane graph on S if and only if n is even and $h(S) \leqslant n/2 + 1$.*

A (geometric) graph on at least $k+1$ vertices is *k-edge-connected* if it is connected and it remains connected whenever $k-1$ edges are removed. The above results hold also for 3-edge-connectivity:

Theorem 3. *The statements of Theorem* 1 *and Corollary* 2 *also hold when* 3*-edge-connectivity is considered instead of* 3*-connectivity.*

If we focus on connecting the points of the set S by a cubic plane graph, without the additional requirement of 3-connectivity, the situation changes substantially. Of course, we need that n, the number of points of S, is even. Our main result in this topic is as follows:

Theorem 4. *Let $n \geqslant 4$ be an even integer. Then, we have*:

(i) *Any set S of n points in general position in the plane satisfying $h(S) \leqslant 3n/4$ admits a cubic* 2*-connected plane graph on S.*
(ii) *If h is an integer such that $3n/4 < h < n-1$, then among sets S of n points in general position with $h(S) = h$, at least one set admits a cubic* 2*-connected plane graph on S and at least one set admits no cubic plane graph on S.*
(iii) *Sets S of n points with $h(S) \geqslant n-1$ admit no cubic plane graph on S, with the only exception the case $|S| = n = 4$ with $h(S) = n - 1 = 3$.*

Section 2 contains the proofs of Theorems 1 and 3. Corollary 2 is an immediate consequence of Theorem 1. Section 3 contains the proof of Theorem 4.

2. Triconnected plane graphs

2.1. Lower bounds

Let S be a set of n points in general position in the plane. Let us see that any 3-edge-connected plane graph G on S contains at least $\max\{\lceil 3n/2\rceil, n+h-1\}$ edges. This gives the lower bound in the 3-edge-connectivity version stated in Theorem 3 and therefore also the bound in Theorem 1, since every 3-connected graph is 3-edge-connected.

Since any vertex of a 3-edge-connected graph has degree at least three, the graph G has at least $\lceil 3n/2\rceil$ edges. It remains to show that G has at least $n+h-1$ edges.

Since G is 3-edge-connected, the boundary of the outer face of G is formed by a closed trail W. The trail W visits all the h extremal points of S, and therefore it contains at least h edges.

Let $G' = G - E(W)$ be the plane graph obtained from G by deleting the edges of W. It suffices to prove that G' is connected, since then G has $|E(W)| + |E(G')| \geqslant h + (n-1)$ edges.

Any two faces of G meet in at most one edge, as otherwise the removal of any two of their common edges would disconnect G. Now, let $p, q \in S = V(G')$, and let W_{pq} be a walk in G connecting p and q and containing the smallest possible number of edges of W. Suppose that W_{pq} contains some edge e in W. Let F be the inner face of G containing e. Then e may be replaced in W_{pq} by the other edges of F. This gives a walk from p to q with a smaller number of edges in W, contradicting the choice of W_{pq}. Therefore, W_{pq} must not contain an edge of W. It follows that $G' = G - E(W)$ is connected, as required.

2.2. Construction of a 3*-connected plane graph on S with the minimum number of edges*

In this subsection we construct a 3-connected plane graph G on S with $\max\{\lceil 3n/2\rceil, n+h-1\}$ edges when S is not in convex position and $|S| = n \geqslant 4$. Together with the lower bound proved in the preceding subsection and with the remarks on algorithms described at the end of this section, this proves Theorems 1 and 3.

2.2.1. Preparatory results

Let S be a set of $n \geqslant 4$ points in general position in the plane, not in convex position. Our construction will always give a plane graph G on S such that every two consecutive vertices of H are connected by an edge. We denote the cycle formed by these h edges by $C = C(S)$. The following lemma describes a class of plane graphs G' on S, for which the plane graph $G = G' \cup C$ is 3-connected.

Lemma 5. *Let G' be a connected plane graph on S, in which the degree of each point of H is* 1*. Furthermore, suppose that G' verifies the following property*:

(3P) *For every point $v \in I = S \setminus H$, the graph G' contains three disjoint paths from v to distinct points of H (i.e., any two of these paths meet only in v).*

Then, the plane graph $G = G' \cup C$ is 3*-connected.*

Proof. Let $u, u' \in V(G) = S$. If u and u' lie both on C then $G - \{u, u'\}$ is connected because G' is connected and u and u' have degree 1 in it. In all other cases it suffices to show that in $G - \{u, u'\}$, each vertex $v \in I$ is connected by a path to a vertex lying on C. At least one of the three disjoint paths from v given by property (3P) contains neither u nor u', and therefore it lies in $G - \{u, u'\}$. Thus, G is 3-connected. □

A *plane tree* is a plane graph whose underlying graph is a tree. The notions *geometric tree* and *geometric* (or *plane*) *forest* or *star* are defined analogously. Notice that, clearly, for a plane tree G' with H as its set of leaves and no vertex of degree 2, the condition (3P) from Lemma 5 holds.

We construct the required 3-connected plane graph G on S by building a plane graph G' satisfying the assumptions of Lemma 5 and then taking $G := G' \cup C$. In the construction of G' we use the following result:

Theorem 6. *(See Tamura and Tamura [20].) Given a set S of n points $p_1, \ldots, p_n$ in general position in the plane and n positive integers $d_1, \ldots, d_n \geqslant 1$ satisfying $\sum_{i=1}^{n} d_i = 2n - 2$, we can construct a plane tree on S, such that the degree of p_i is d_i for $i = 1, \ldots, n$. In fact, any shortest geometric tree satisfying these degree conditions is plane* (*we measure the length of a geometric graph by the sum of lengths of its edges*)*.*

In the construction of G' we distinguish two cases. If $n/2 + 1 \leqslant h < n$ then G' consists of a plane tree which can be found quite easily using Theorem 6. If $h < n/2 + 1$ then the construction is much more complicated, and we prove next two auxiliary lemmas that we are using for that case. The first of them strengthens Theorem 6 for certain special degree conditions:

Lemma 7. *Let S_1 be a subset of S with $t \geqslant n/2 + 1$ points, and let $S_3 = S \setminus S_1$. Suppose that S_1 is partitioned into three disjoint classes H_1, H_2, H_3, each of size at most $n/2$. Then we can construct a plane forest on S as a set of $k := t - n/2 \geqslant 1$ plane trees $T_1, \ldots, T_k$ satisfying the following two conditions*:

(1) *Each point of S_i has degree i for $i = 1, 3$, and*
(2) *each T_j has leaves in at least two of the classes H_1, H_2, H_3.*

Proof. First we prove that there are $k := t - n/2$ geometric trees $T_1, \ldots, T_k$ verifying conditions (1) and (2), but the geometric forest $T_1 \cup T_2 \cup \cdots \cup T_k$ can have crossings. To this aim, take as $T_1, \ldots, T_{k-1}$ the $k - 1$ edges of a matching, each edge linking points of different classes H_i, in such a way that the $t - 2(k - 1) = n - t + 2$ unmatched points of S_1 lie in at least two different classes. This matching can be formed, for example, by repeatedly choosing an edge linking two points belonging to the two classes with the largest numbers of remaining (unmatched) points. Finally, by Theorem 6, T_k will be a geometric tree on the remaining points, such that the degree 3 is given to the $n - t$ points of S_3 and degree 1 to the remaining $t - 2(k - 1) = n - t + 2$ points of S_1. The geometric forest $T_1 \cup T_2 \cup \cdots \cup T_k$ verifies conditions (1) and (2), but its edges can cross.

Consider all the geometric forests on S formed by k geometric trees verifying the conditions (1) and (2). Choose a shortest geometric forest F_0 among them, i.e. a forest with the smallest total edge length. We claim that F_0 is plane.

Suppose on the contrary that two edges in F_0, ab and cd, cross. Since a shortest forest is formed by shortest trees, and a shortest tree is plane by Theorem 6, these two edges have to belong to two different trees. Let T^a, T^b, T^c, T^d be the four components (subtrees) obtained from these two trees by deleting the edges ab and cd, and denoted in such a way that T^x contains x for $x = a, b, c, d$. Then, replacing the edges ab and cd by the edges ac and bd in F_0, we obtain a shorter forest with k trees, where all the vertices have the same degrees as in F_0. This is possible only if some of the two new trees does not verify condition (2). Suppose, for example, that H_1 contains all the leaves of the tree formed by the edge ac and by the subtrees T^a and T^c. Then, since F_0 satisfies condition (2), both T^b and T^d contain some points of $H_2 \cup H_3$. Consequently, if we replace the edges ab and cd by the edges ad and bc in F_0, we obtain a shorter forest verifying conditions (1) and (2), a contradiction. □

The following lemma will be used in the verification of the 3-connectivity of the final plane graph G.

Lemma 8. *If a plane graph G' satisfies the assumptions of Lemma* 5 *then it also satisfies the following strengthening of property* (3P):

(3P′) *For every $v \in I$ and for every $w \in H$, the graph G' contains three disjoint paths from v to distinct points of H* (*any two of these paths meet only in v*)*, such that one of the paths ends in w.*

Proof. Suppose we are given three vertex-disjoint paths P_1, P_2, P_3 in G' from v to distinct points of H. Since G' is connected, it also contains a path P from w to v. We traverse the path P from w to v. Let x be the first point on P, lying on one of the paths P_1, P_2, and P_3. Without loss of generality, the point x lies on P_1. We change the path P_1 so that its subpath from v to x is kept and extended along P from x to w. Then the three paths satisfy (3P′). □

2.2.2. Construction in the case $n/2 + 1 \leqslant h < n$

First, we describe our construction in case $n/2 + 1 \leqslant h < n$. Assign degree $d_i = 1$ to the h points of H, degree $d_i = 3$ to arbitrary $n - h - 1$ points of I, and degree $d_i = 2h - n + 1 \geqslant 3$ to the remaining point of I. Then, $\sum_{i=1}^{n} d_i$ is equal to $h \cdot 1 + (n - h - 1) \cdot 3 + 1 \cdot (2h - n + 1) = 2n - 2$. By Theorem 6, we can construct a plane tree, G', with the points of S having the prescribed degrees d_i. Clearly, G' satisfies the assumptions of Lemma 5, since it is a plane tree with no vertex of

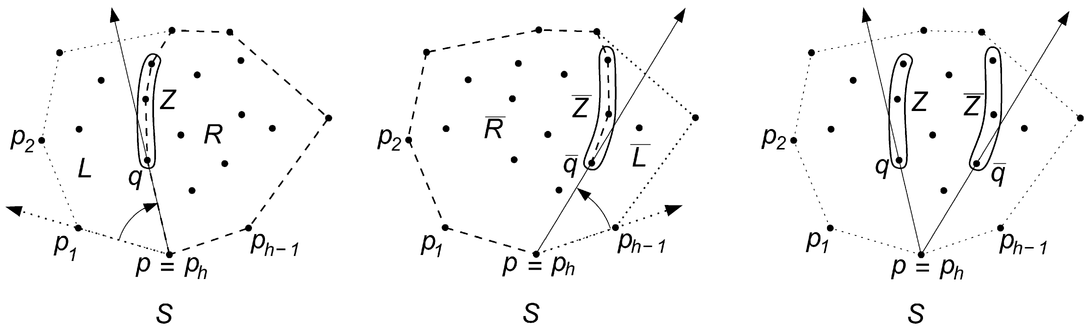

Fig. 2. Left: The point q and the sets L, R, and Z. Middle: The point $\bar{q}$ and the sets $\overline{R}$ and $\overline{Z}$. Right: The points q and $\bar{q}$ and the sets Z and $\overline{Z}$.

degree 2. Thus, by Lemma 5, the plane graph $G = G' \cup C$ is 3-connected. Also, it has $(n-1)+h = \max\{\lceil 3n/2\rceil, n-1+h\}$ edges, as required.

2.2.3. Construction in the case $h < n/2+1$ for n even

When $h < n/2+1$, the construction is much more complicated than in the previous case. We first consider the subcase in which $|S| = n$ is even; the construction for odd n is given in the next subsection. Notice that $n \geqslant 6$ because $h \geqslant 3$ for any set S.

Since $\max\{\lceil 3n/2\rceil, n-1+h\} = 3n/2$ in this subcase, we need to construct a 3-connected cubic plane graph G on S. According to Lemma 5, it suffices to construct a connected plane graph G' on S, such that the points of H have degree 1, the points of $I = S \setminus H$ have degree 3, and G' verifies property (3P) of three disjoint paths given in that lemma.

Our construction of G' is (partially) inductive, meaning that in some cases we use the construction for a smaller set. The plane graph G' is constructed as a union of two plane graphs lying on different sides of a line pq defined below.

We denote the points of H in the clockwise order by $p_1, p_2, \dots, p_h$. For simplicity of notation, the point p_h is also denoted by p. For a point $x \in I$, let $L(x)$ denote the set of points of S lying in the closed halfplane to the left of the ray px. Consider the ray pp_1 and rotate it clockwise around p. We stop the rotation when we reach a point $q \in I$ such that the set $L := L(q)$ satisfies $|L \cap I| = |L \cap H| - 1$. In other words, we stop the rotation when the number of interior points to the left of px, including q, is the same as the number of extremal points to the left of px, excluding p. See Fig. 2 (left). The existence of q easily follows from the general position of S and from the assumption $h < n/2+1$.

Let R denote the set of points of S lying in the closed halfplane to the right of the ray pq. Since $p_{h-1} \notin L$, then $|L \cap H| \leqslant h-1$ and $h - (|L \cap H| - 1) \geqslant 2$. Therefore, the assumption $h < n/2+1$ implies that R must contain al least two interior points different from q. Further, let Z be the set of points of I lying on the boundary of the convex hull of the points of R. See Fig. 2 (left). Let z and r be the sizes of Z and R, respectively. Notice that q belongs to Z, that q, p, p_{h-1} belong to R and that $r \geqslant 5$.

In our construction we need that $z < r/2$. In general, the inequality $z < r/2$ may be false. However, an analogous inequality is then true if we rotate the ray pp_{h-1} counterclockwise around p instead of rotating pp_1 clockwise. Let $\bar{q}, \bar{L}, \bar{R}, \bar{Z}, \bar{r}, \bar{z}$ be the point, the three sets, and the two numbers obtained when we do the same construction as above, but rotating pp_{h-1} counterclockwise. See Fig. 2 (middle). Now, $\bar{L}$ is the set of points of S lying in the closed halfplane to the right of the ray $p\bar{q}$ such that $|\bar{L} \cap I| = |\bar{L} \cap H| - 1$.

Lemma 9. *$z < r/2$ or $\bar{z} < \bar{r}/2$.*

Proof. By construction and by $h < n/2+1$, the point $\bar{q}$ lies to the right of the ray pq in S. See Fig. 2 (right).

The two rays pq and $p\bar{q}$ divide the points of $H \setminus \{p\}$ into three classes H_1, H_2, H_3, and the points of I into three classes I_1, I_2, I_3 as shown in Fig. 3 (left), whereas q is put in I_1 and $\bar{q}$ is put in I_3.

From the choice of q and $\bar{q}$ we get $|H_1| = |I_1|$ and $|H_3| = |I_3|$.

We now estimate the size of $Z \cap \overline{Z}$. Let the ray pq cut the set H between the points p_j and p_{j+1}. If $H_2 \neq \emptyset$ then $Z \cap \overline{Z} = \emptyset$; see Fig. 2 (right). If $H_2 = \emptyset$, then clearly, by convexity, all the points of $R \cap H$ ($\overline{R} \cap H$ respectively) lie strictly to the right (left respectively) of any line through two arbitrary points of Z ($\overline{Z}$ respectively). Now, since $p \in (R \cap \overline{R} \cap H)$, if $|Z \cap \overline{Z}| \geqslant 2$, then p should be simultaneously to the right and to the left of a line through two points of $|Z \cap \overline{Z}|$. Hence, $|Z \cap \overline{Z}| \leqslant 1$. See Fig. 3 (right).

Therefore

$$\begin{aligned} z + \bar{z} &= |Z| + |\overline{Z}| = |Z \cup \overline{Z}| + |Z \cap \overline{Z}| \\ &\leqslant \big(|I_1| + |I_2| + |I_3|\big) + 1 = \big(|I_1| + |H_1|\big)/2 + |I_2| + \big(|I_3| + |H_3|\big)/2 + 1 \\ &\leqslant \big(|I_1| + |H_1|\big)/2 + |I_2| + |H_2| + \big(|I_3| + |H_3|\big)/2 + 1 = \big(\big|R \setminus \{p,q\}\big| + \big|\overline{R} \setminus \{p,\bar{q}\}\big|\big)/2 + 1 = r/2 + \bar{r}/2 - 1. \end{aligned}$$

It follows that $z < r/2$ or $\bar{z} < \bar{r}/2$. $\square$

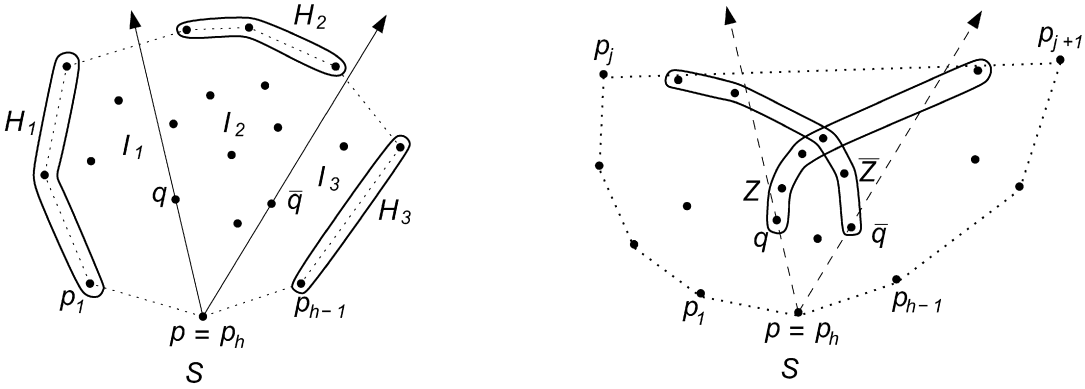

Fig. 3. Left: The classes $H_1, H_2, H_3, I_1, I_2, I_3$. Right: The sets Z and $\overline{Z}$ intersect in at most one point.

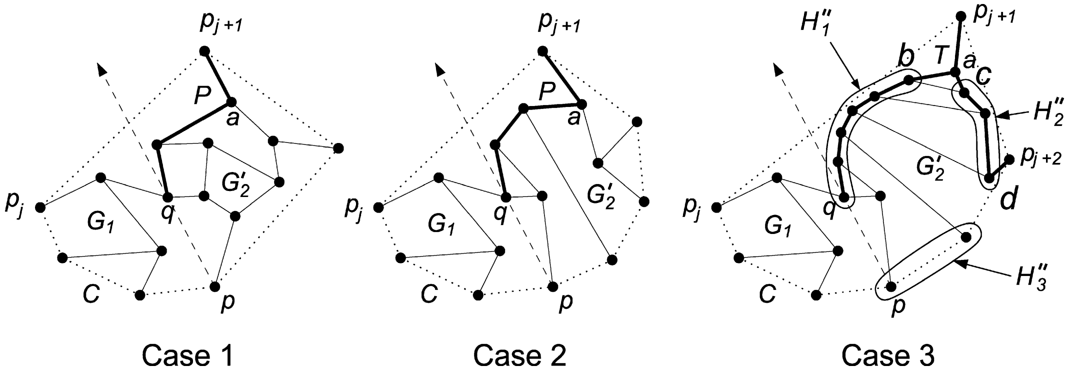

Fig. 4. The 3-connected cubic plane graph $G = C \cup G'$, where $G' = G_1 \cup (G'_2 \cup P)$ or $G' = G_1 \cup (G'_2 \cup T)$ (the circle C is drawn by dotted segments; the two plane graphs G_1 and G'_2 are drawn by thin solid segments and they are separated by the ray pq; the path P or the tree T is drawn in fat in each of the three cases).

By Lemma 9, we can assume that $z = |Z| < r/2$, since otherwise we could consider the partition of S defined by ray $p\bar{q}$ instead of ray pq.

Our plane graph G' will be a union graph of two connected plane graphs G_1 and G_2, where the vertex sets of G_1 and G_2 are $L \setminus \{p\}$ and R, respectively.

We construct G_1 as a plane tree, in which the points of $(L \cap I) \setminus \{q\}$ have degree 3 and all the other points of $L \setminus \{p\}$ have degree 1. We may construct a plane tree G_1 with these degrees according to Theorem 6. See Fig. 4, where G_1 is the tree to the left of the ray pq.

The construction of G_2 is more complicated. The points of $R \cap H$ will have degree 1, the point q will have degree 2, and all the other points of R will have degree 3 in G_2. Then the union graph $G' = G_1 \cup G_2$ has the required degrees and the graph $G = G' \cup C$ is cubic. Thus, we need to construct a connected G_2 with the above degrees such that the graph $G' = G_1 \cup G_2$ satisfies property (3P).

Let the ray pq cut the set H between the points p_j and p_{j+1}. Let $S' := R \setminus \{p_{j+1}\}$, and let $H' \geqslant 3$ be the set of vertices of the convex hull of S'. Note that $Z \subset H'$. By construction, r is odd, hence $|S'| = r - 1$ is even. In the construction of G_2 we distinguish the following three cases:

Case 1: $|H'| < r/2$,
Case 2: $|H'| > r/2$ and $|H' \setminus H| < r/2$,
Case 3: $|H' \setminus H| > r/2$.

Case 1 can only appear if $n \geqslant 8$, since $H' \geqslant 3$ and hence $r \geqslant 7$. If $n = 6$, then necessarily $h = 3$ and it can easily be checked that $r = 5$. So, when $n = 6$, only cases 2 or 3 can appear.

We start with Case 1. Using the inductive hypothesis, we can construct a connected plane graph G'_2 on S' verifying property (3P), such that the degrees are 1 at points of H' and 3 at the other points of S'. Let $a \in H'$ be the counterclockwise neighbor of p_{j+2} in H' (possibly $a = q$). Connect the points on $H' \setminus H$ by a path and add the edge ap_{j+1} to this path. Let P be the obtained path from q to p_{j+1}. See Fig. 4 (left). Let G_2 be the union graph $G'_2 \cup P$. The points in G_2 have the required degrees and the union graph $G' = G_1 \cup G_2$ is connected. It remains to show that G' verifies property (3P).

In G', property (3P) is satisfied for vertices of degree 3 of G_1, because G_1 is a tree, and a path finishing in q can be continued through P until p_{j+1} is reached. In the same way, for the vertices on $H' \setminus H$, there is a path along P to q that can be continued in G_1 to p_j (say), another path in G'_2 to p (say), and a third path along P to p_{j+1}, therefore we get three disjoint paths ending in points of H. Finally, if v is a vertex of degree 3 in G'_2 then by induction, G'_2 contains three disjoint paths from v to leaves of G'_2, and we can suppose by Lemma 8 that one of them finishes in the point p. Therefore, at most two of the paths finish in points on P. But then, we can extend one of these paths along P to p_{j+1} and the other one to q and then in G_1 to p_j, hence obtaining again three disjoint paths in G' arriving to points of H. This concludes Case 1.

We now consider Case 2, where $|H'| > r/2$ and $|H' \setminus H| < r/2$. We divide the points on H' into two classes, $H'_1 := H' \setminus H$ and $H'_2 := H' \cap H$. By construction, $n = |L \cap I| + |L \cap H| - 1 + r - 1 = r + 2|L \cap H| - 3$ and $(L \cap (H \setminus p)) \cup H'_2 \cup p_{j+1} \subseteq H$. This implies $|L \cap H| - 1 + |H'_2| + 1 \leqslant |H| < \frac{r+2|L \cap H|-3}{2} + 1$, hence $|H'_2| < \frac{r-1}{2} = \frac{|S'|}{2}$. Therefore, we may apply Lemma 7 on S' (taking the points of H' as the vertices of degree 1, partitioned into the three classes H'_1, H'_2, and $H'_3 := \emptyset$). We obtain a plane forest G'_2 on S' such that each of its trees has leaves both in H'_1 and in H'_2. See Fig. 4 (middle). If we define a path P as in Case 1, then we can check similarly as in Case 1 that the plane graph $G_2 := G'_2 \cup P$ verifies all the required properties.

Finally, we consider Case 3, where the number of points of $H' \setminus H$ exceeds $r/2$. Note that $r \geqslant 5$. Let a be the point of $H' \setminus H$ such that the interval of $H' \setminus H$ between q and a contains exactly $(r-1)/2$ points (including the endpoints). So, there is at least one more point after a in clockwise order on $H' \setminus H$. Since $z = |Z| < r/2$, we can link p_{j+1} with a without crossing the boundary of the convex hull of S'. Consider the set of points $S'' := S' \setminus \{a, p_{j+2}\}$, and let H'' be the set of points on the boundary of its convex hull. Let b be the counterclockwise neighbor of a in H'. See Fig. 4. We partition the set H'' into three parts: H''_1 with the points from q to b, H''_2 with the other points of $H'' \cap I$, and H''_3 with the points of $H'' \cap H$ (possibly $H''_3 = \emptyset$). By the choice of a and b, the set H''_1 contains exactly half of the points of S'' and H''_2 is not empty. Thus, $|H''| \geqslant (r-3)/2 + 1$ and we can apply Lemma 7 to the set S'', taking the points of H'' as the vertices of degree 1, divided into the three classes H''_1, H''_2 and H''_3. We obtain a plane forest G'_2 on S''.

We denote the first and the last point in the clockwise order of H''_2 by c and d, respectively (possibly $c = d$). See Fig. 4 (right). Let us connect consecutively the points of H''_1 by a path, add three edges connecting a with the points b, c, and p_{j+1}, connect consecutively the points of H''_2 by a path, and finally add the edge dp_{j+2}. If we denote by T the tree with this set of added edges, we can check similarly as in the previous cases that the graph $G_2 := G'_2 \cup T$ verifies all the required conditions. Note that G_2 is plane by convexity and by the choice of a. This finishes the construction in Case 3.

2.2.4. Construction in the case $h < n/2 + 1$ for n odd

If the number of points of the set S is an odd number n, it is possible to mimic the preceding construction, now for an odd number of points. Another possibility is to take the preceding construction for $S \setminus \{p\}$, where $p \in S$, and then introduce some adjustment to obtain the required construction for the whole set S. We present a construction proceeding in the latter approach.

For a point $p \in H$, we use the notations $S_p := S \setminus \{p\}$, H_p is the set of vertices of the convex hull of S_p, and $N_p := H_p \setminus (H \setminus \{p\})$. Clearly, the points of N_p lie in the interior of the triangle T_p having vertices in p and in the two neighbors of p in the cyclic order of H (see Fig. 5). It follows that two sets N_p and $N_{p'}$ may intersect only if p and p' are neighbors in the cyclic order of H. Moreover, the intersection $N_p \cap N_{p'}$ has size at most one, since it may contain only the point of I which is (strictly) closest to the line pp'. The sum of the sizes of the h sets $N_p, p \in H$, is therefore at most $|I| + h = n$. It follows by an averaging argument that one of the sets $H_p = N_p \cup (H \setminus \{p\})$ has size at most $\lfloor \frac{n}{h} + (h-1) \rfloor = \lfloor \frac{n}{2} + 1 + \frac{(2h-n)(h-2)}{2h} \rfloor \leqslant \lfloor \frac{n}{2} + 1 + \frac{1 \cdot (h-2)}{2h} \rfloor = \frac{n+1}{2}$.

Now, fix $p \in H$ such that $|H_p| \leqslant \frac{n+1}{2}$. Then we may use the construction for the even case on $S_p = S \setminus \{p\}$. We obtain a 3-connected cubic plane graph G_0 on S_p, such that any two consecutive vertices of the convex hull of S_p are connected by an edge.

Let $N_p = \{x_1, x_2 \dots, x_t\}$, let x_0 and x_{t+1} be the neighbors of p in H, and let $x_0, x_1, \dots, x_t, x_{t+1}$ appear in this order in H_p. Further, let G_1 be the plane graph obtained from G_0 by deleting the edge x_0x_1 and adding the two edges x_0p and x_1p. Since G_0 is 3-connected, the only vertex cut of size 2 in G_1 is obviously $\{x_0, x_1\}$. The graph $G_1 - \{x_0, x_1\}$ has exactly two components, one of them consisting of a single vertex p. Therefore, if we add any edge pq, $q \in S \setminus \{p, x_0, x_1\}$, to G_1, we obtain a 3-connected graph on S with $\frac{3(n-1)}{2} + 2 = \lceil \frac{3n}{2} \rceil$ edges. It remains to show that there is a $q \in S \setminus \{p, x_0, x_1\}$ such that

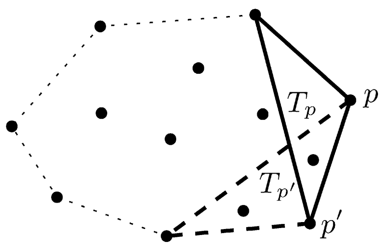

Fig. 5. Two triangles T_p and $T_{p'}$.

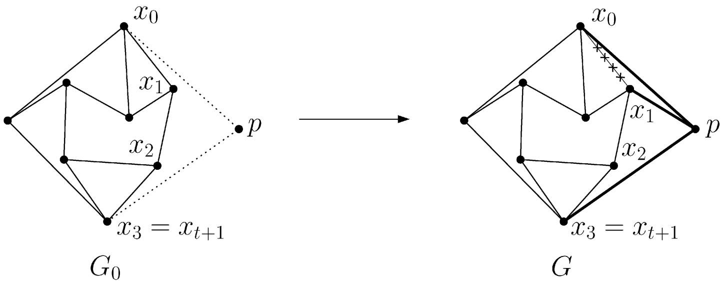

Fig. 6. Construction in case $N_p \neq \emptyset$.

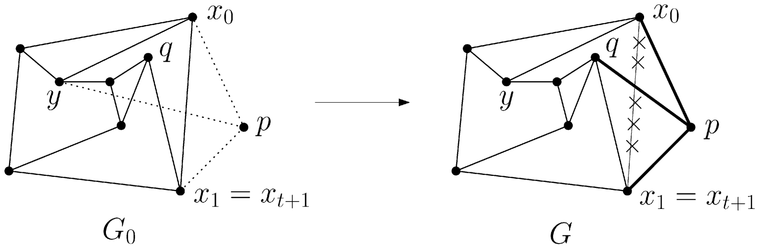

Fig. 7. Construction in case $N_p = \emptyset$.

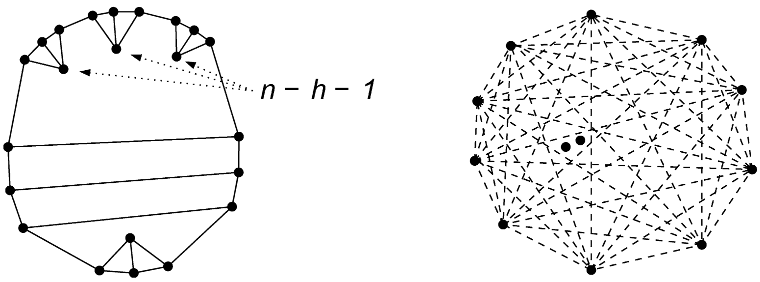

Fig. 8. Left: A point set which allows a cubic 2-connected plane graph. Right: A point set which does not allow a cubic plane graph.

the open segment pq crosses no edge of G_1. If $N_p \neq \emptyset$ then we may put $q = x_{t+1}$ (see Fig. 6). Suppose now that $N_p = \emptyset$. Let $x_0 y$ be the edge of G_1 incident to x_0 and not lying on the boundary of the convex hull of S_p. Consider the set A of points of $S \setminus \{p, x_0\}$ lying in the triangle $px_0 y$. We have $y \in A$, thus $A \neq \emptyset$. Let q be the point of A minimizing the angle $\angle x_0 pq$ (see Fig. 7). Then, the segment pq intersects no edge of G_1, as required. This concludes the construction in the case $h < n/2 + 1$ for n odd.

We remark that also in this case, we have constructed a plane graph containing all the h edges of the cycle C.

2.2.5. Remarks on the algorithmic aspects

While the main goal of the preceding results is to characterize the conditions in which point sets admit 3-connected plane graphs, it is worth noticing that all the proofs used so far to obtain the previous results are constructive; in other words, they provide an effective method for building an actual 3-connected plane graph on S. This method is polynomial, because the process used in Theorem 6 and Lemma 7 to eliminate crossings terminates in a polynomial number of steps ($O(n^3)$ steps, see [21]). This proves our claims on this regard.

Let us mention that instead of eliminating crossings, in [3] they use what they call *hull trees* as a key structure to store information, which allows building the plane tree given in Theorem 6 with $O(n \log n)$ complexity. However, for our problem, there are several additional algorithmic issues that should be addressed, depending on the different cases, a task that we leave for future research as some of the cases are unclear to us. Hence, a better complexity for the overall construction may be possible, but we do not pursue this possibility here.

3. Cubic plane graphs

In this section we prove Theorem 4.

We prove first part (ii). Fig. 8 (left) depicts a cubic 2-connected plane graph on a particular set S for any given parameters $h, n, 3n/4 < h < n - 1$.

On the other hand, let S be a set of $n \geqslant 4$ points in general position such that $3n/4 < h(S) < n-1$ and such that all the points of $I = I(S)$ lie in the same cell of the arrangement of the $\binom{h(S)}{2}$ lines determined by H; see Fig. 8 (right). We now prove that S admits no cubic plane graph.

Let G be a plane graph on S. Suppose that G is cubic. Then, since $|H| > 3|I|$, there is an edge pp' in G connecting two non-consecutive points $p, p' \in H$. Let H' be the non-empty set of points of H separated from I by the line pp'. If we choose the edge pp' in such a way that H' is as small as possible, then all the points of H' have degree at most 2 in G, a contradiction. Therefore, there is no cubic plane graph on S. This finishes the proof of part (ii).

We now prove part (iii). If $h(S) = n-1$ and $|S| = n > 4$, then we can show in the same way as above that S admits no cubic plane graph. If $h(S) = n$ then S is in convex position and any plane triangulation of S contains vertices of degree two and therefore S admits no cubic plane graph.

What is left now is to prove part (i). Let S be a set with n even and with $h \leqslant 3n/4$. If $h \leqslant n/2+1$ then we may use the construction from the previous section resulting in an even 3-connected plane graph. It remains to construct a cubic plane graph on S when $n/2+1 < h \leqslant 3n/4$. Similarly as in the previous section our construction gives a plane graph G on S containing all the h edges of $C = C(S)$ connecting consecutive pairs of points of H. Thus, it certainly suffices to find a plane forest G' on S satisfying the following three conditions:

(C1) The points of H have degree 1,
(C2) the points of I have degree 3, and
(C3) no edge connects two points of degree 1.

Note that conditions (C1) and (C3) guarantee that G' contains no edge of $C(S)$.

If we allow crossings, then a geometric forest on S satisfying conditions (C1)–(C3) may be constructed, for example, by partitioning the points of S according to their prescribed degrees into $h - n/2 - 1$ (geometric) stars $K_{1,3}$ and a (geometric) tree having $3n/2 - 2h + 1 \geqslant 1$ vertices of degree 3 and $3n/2 - 2h + 3 \geqslant 3$ leaves.

Now, let G' be a geometric forest on S satisfying conditions (C1)–(C3) and having the minimum possible total edge length. Let us show that G' is plane. Suppose on the contrary that two edges in G', ab and cd, cross. By Theorem 6 and by the minimality of G', each tree in G' is plane. Therefore, the crossing edges ab and cd lie in different trees. Due to condition (C3), we may assume that both a and c have degree 3. Then, if we replace the edges ab and cd by the edges ad and bc, we obtain a geometric forest on S satisfying conditions (C1)–(C3) and having a smaller total edge length, a contradiction. Hence, G' is plane.

Lastly, note that G' does not have to be 3-connected. For instance, consider the example in Fig. 1 (right).

4. Final remarks

Improving on the efficiency of the algorithms described at the end of Section 2 is one of the natural open problems left for future research, as is the extension of our results when 4-connectivity is considered instead of 3-connectivity.

On the other hand, it has also been pointed to us by Emo Welzl that it would be interesting to explore what bounds can be obtained in Theorem 1, if we allow only those 3-connected plane graphs on S in which every inner face is a convex polygon and their union is also a convex polygon. Note that any plane graph may be changed to a graph with such faces by adding an appropriate set of edges. Thus, the existence is guaranteed for any S in general position that is not in convex position. Plane graphs with convex faces are intensively studied and it is worth mentioning in this context that if the upper envelope of a 3-dimensional convex polytope is a single face, then the orthogonal projection of the 1-skeleton of the polytope to the xy-plane is a 3-connected plane graph with convex faces.

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